The euclidean ball and the polydisc

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the Riemann mapping theorem

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- The Riemann mapping theorem says that there is bi-holomorophic map between the domain Ω and the open unit disc $\mathbb{D}$.
- Thus the set of holomorphic functions on Ω can be identified with holomorphic functions on $\mathbb{D}$ using the bi-holomorphic Riemann map between these two sets.

the ball and the polydisc

- Let $\mathbb{B}_m := \{z := (z_1, \dots z_m) \in \mathbb{C}^m \mid |z_1|^2 + \cdots + |z_m|^2 < 1\}$ be the Euclidean ball in $\mathbb{C}^m$. Also, let $\mathbb{D}^m := \{z := (z_1, \dots z_m) \in \mathbb{C}^m \mid |z_1|, \dots, |z_m| < 1\}$ be the polydisc.

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- Both the ball $\mathbb{B}^m$ and the polydisc $\mathbb{D}^m$ are simply connected: These are the unit balls in the ℓ_2 and ℓ_∞ norms, respectively. Therefore, in particular convex. Hence both of these are simply connected. It is natural to ask if they are bi-holomorphic.

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- The map ϕ is said to be bi-holomorphic if it admits an inverse, that is, if there is a holomorphic map $\psi : \mathbb{D}^m \to \mathbb{B}_m$ such that $\psi \circ \phi : \mathbb{B}_m \to \mathbb{B}_m$ is the identity map.

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- We ask if there is such a bi-holomorphic map between $\mathbb{B}_m$ and $\mathbb{D}^m$, $m > 1.$

Schwarz lemma

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  If $f : \mathbb{B} \to \mathbb{D}$ is any holomorphic function with $f(0) = 0$, then for any $v \in \mathbb{B}$, we have $|Df(0) \cdot v| \leq 1$, where

$$Df(0) \cdot v = v_1 \partial_1 f(0) + \cdots + v_m \partial_m f(0).$$

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- For the verification of this inequality, pick any $w\in\mathbb{B},$ and let $\Lambda_w:\mathbb{D}\to\mathbb{B}$ be the function $\Lambda_w(z)=zw.$
- We have that $f\circ\Lambda_w:\mathbb{D}\to\mathbb{D}$ and $f\circ\Lambda_w(0)=0.$ Therefore, by the Schwarz lemma,

$$|D(f\circ\Lambda_w)(0)| = |Df(\Lambda_w(0))D\Lambda_w(0)| = |Df(0)\cdot w|\le 1$$

Carath'eodory norm

- For $\Omega \subset \mathbb{C}^m$, $v \in \mathbb{C}^m$, and $w \in \Omega$, the Carathéodory norm $C_{\Omega,w}(v)$ of the vector v at w is defined to be the supremum:

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- The Schwarz lemma for $\mathbb{B}$ shows that $\mathbb{B} \subseteq \{v \mid C_{\mathbb{B},0}(v) < 1\}$. To check the inclusion, the other way round, we observe that if $\ell : \mathbb{B} \to \mathbb{C}$ is a linear functional with $\ell(\mathbb{B}) \subseteq \mathbb{D}$, then $D\ell(0) = \ell$ and hence if we pick v such that $C_{\mathbb{B},0}(v) < 1$, then $\|v\|_{\mathbb{B}} < 1$. Thus we have $\mathbb{B} \supseteq \{v \mid C_{\mathbb{B},0}(v) < 1\}$ proving the claim.

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- Thus $C_{\mathbb{B},0}(v) = \|v\|_{\mathbb{B}}.$

contractivity

- Suppose that $\varphi : \mathbb{B} \to \mathbb{B}'$ is a holomorphic function with $\varphi(0) = 0$. Then

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- proof: Set $v' = D\varphi(0) \cdot v$. We have

$$\begin{aligned} C_{\mathbb{B}',0}(v') &= \sup\{|Df(0) \cdot v'| \mid f : \mathbb{B}' \to \mathbb{D}, \text{holomorphic}, f(0) = 0\} \\ &= \sup\{|Df(0) D\varphi(0) \cdot v| \mid f : \mathbb{B}' \to \mathbb{D}, \text{holomorphic}, f(0) = 0\} \\ &= \sup\{|D(f \circ \varphi)(0) \cdot v| \mid f : \mathbb{B}' \to \mathbb{D}, \text{holomorphic}, f(0) = 0\} \\ &\leq \sup\{|Dg(0) \cdot v| \mid g : \mathbb{B} \to \mathbb{D}, \text{holomorphic}, g(0) = 0\} \\ &= C_{\mathbb{B},0}(v) \end{aligned}$$

Or in other words, $D\varphi(0) : \mathbb{C}^m \to \mathbb{C}^n$ is a contraction.

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implying $D\psi(0) = D\varphi(0)^{-1}.$ Thus we have

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- It follows that both $D\varphi(0)$ as well as $D\varphi(0)^{-1}$ are contractions. Hence (easy exercise) we have that $\|D\varphi(0) \cdot v\|_{\mathbb{B}'} = \|v\|_{\mathbb{B}}$.

parallelogram law

- Now, suppose that there is a bi-holomorphic map $\varphi : \mathbb{B}_m \to \mathbb{D}^m$. Let $w = \varphi(0) \in \mathbb{D}^m$. Composing with the Möbius maps $\varphi_1, \dots, \varphi_m$ of the unit disc with $\varphi_i(w_i) = 0$, we can assume, without loss of generality that $\varphi(0) = 0$. We have shown that the linear map $D\varphi(0)$ must be then an isometry between the two spaces $(\mathbb{C}^m, \|\cdot\|_2)$ and $(\mathbb{C}^m, \|\cdot\|_\infty)$.

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- No isometry between these two spaces can exist since one of them comes from an inner product, therefore satisfies the parallelogram law

$$2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2,$$

while the other doesn't.

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- Suppose that Γ is a linear isometry between $(\mathbb{C}^m, \|\cdot\|_2)$ and $(\mathbb{C}^m, \|\cdot\|_\infty)$. Then any pair of vectors in $(\mathbb{C}^m, \|\cdot\|_\infty)$ must also obey the parallelogram law.

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- Suppose that Γ is a linear isometry between $(\mathbb{C}^m, \|\cdot\|_2)$ and $(\mathbb{C}^m, \|\cdot\|_\infty)$. Then any pair of vectors in $(\mathbb{C}^m, \|\cdot\|_\infty)$ must also obey the parallelogram law.
- The pair of vectors $u := (1, 0 \ldots, 0)$ and $v := (0, 1, 0 \ldots, 0)$ in $\mathbb{C}^m$ evidently violate the parallelogram law leading to a contradiction.

Thank You!